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# Size effects in non-linear heat conduction with flux-limited behaviors

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Size effects are discussed for several non-linear heat conduction models with flux-limited behaviors, including the phonon hydrodynamic, Lagrange multiplier, hierarchy moment, nonlinear phonon hydrodynamic, tempered diffusion, thermon gas and generalized nonlinear models. For the phonon hydrodynamic, Lagrange multiplier and tempered diffusion models, heat flux will not exist in problems with sufficiently small scale. The existence of heat flux needs the sizes of heat conduction larger than their corresponding critical sizes, which are determined by the physical properties and boundary temperatures. The critical sizes can be regarded as the theoretical limits of the applicable ranges for these non-linear heat conduction models with flux-limited behaviors. For sufficiently small scale heat conduction, the phonon hydrodynamic and Lagrange multiplier models can also predict the theoretical possibility of violating the second law and multiplicity. Comparisons are also made between these non-Fourier models and non-linear Fourier heat conduction in the type of fast diffusion, which can also predict flux-limited behaviors.



## 1. Introduction

As a phenomenological model, Fourier's law of heat conduction has been proved by numerous experiments and widely used in engineering. It describes a constitutive relation between the temperature gradient and heat flux

$$\boldsymbol{q} = -\lambda \nabla T, \tag{1}$$

where $\boldsymbol{q}$ is the heat flux, $\lambda$ is the thermal conductivity and $T$ is the temperature. In statistical mechanics, Fourier's law has been derived approximately through several given theoretical assumptions, which also implies its possible restrictions, i.e., near-equilibrium region. Especially in nanoscale heat transport [1–6], the effects of far-from-equilibrium can play an important role because the characteristic size can be comparable to the mean free path of heat carriers. The non-Fourier effects in nanoscale can be classified into three types [7]: relaxation, nonlocality and nonlinearity. Relaxation in heat conduction is first introduced by the Cattaneo–Vernotte (CV) model [8,9], whose hyperbolic governing equation predicts a finite wave velocity of heat propagation. It should be noted that in non-linear Fourier heat conduction [12,13], there also exist hyperbolic or wave-like characteristics. For instance, fast (superfast) diffusion [14], where $\lambda = \lambda(T) \propto T^{-\alpha}$ $(0 < \alpha < 2$ is a constant), also has the travelling wave solution $T(x,t) = T_1(x - Ut) + T_2(x + Ut)$ with a finite wave velocity $U$. Thus, hyperbolic or wave-like characteristics are not enough to distinguish non-Fourier relaxation models and Fourier's law. In the spirit of relaxation, the CV model has been generalized to different non-Fourier models, i.e., the Jeffrey model [10,11], a linear superposition of the CV and Fourier heat conductions. The constitutive relations of these relaxation models can usually be summarized as memory behaviors [10,11], where the heat flux is depended on the integrated history of the temperature gradient. Different constitutive models can be given through different choices of the integral kernels. Most of the memory kernels are exponential type or Dirac delta function (or their linear superposition) [10,11]. Power-law kernels can also be applied, which will lead to fractional differential operators [15,16]. The hyperbolic heat conduction models, i.e., the CV model, might predict non-positive values of the absolute temperature which seems unphysical. Recently, hyperbolic heat conductions paired with this behavior have been further discussed by introducing a new class of stochastic processes [17,18], generalized Poisson–Kac processes. The nonlocal and nonlinear effects are mainly found in the models related to phonon hydrodynamics [19,20]. Most of these models are on the basis of phonon Boltzmann transport equation and relaxation approximations, while the thermon gas model [21–24] takes a different method, which considers Einstein's mass–energy relation in phonon hydrodynamics. The second spatial derivatives of the heat flux including $\nabla^2\boldsymbol{q}$ and $\nabla(\nabla \bullet \boldsymbol{q})$ [19,20] are the most common nonlocal terms. For steady-state cases, the relaxation

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terms will disappear as $\frac{\partial}{\partial t}=0$ but nonlocality might still exist due to $\nabla^2\boldsymbol{q}\neq\boldsymbol{0}$, which means that the nonlocal models would not reduce to Fourier's law.

In contrast with relaxation and nonlocality, nonlinearity, which might be unignorable in nanoscale heat transport, is not much studied [25–29]. The nonlinear effects predict significant and interesting phenomenon, i.e., flux-limited behavior [29], where the heat flux tends to a finite upper bound with the temperature gradient increasing. The flux-limited behaviors in heat conduction caused by nonlinearity have been well discussed and reviewed by Guo et al. [25]. They have summarized the nonlinear models with flux-limited behaviors into three categories according to their theoretical foundations: phonon hydrodynamics, nonequilibrium thermodynamics, and phenomenological methods. These models expressed by the local temperature and heat flux distributions aim at providing constitutive relations for nanoscale heat transport. However, from the viewpoint of physics, it is obvious that they cannot be applied to the heat conduction problem with arbitrarily small size because the definitions of the local temperature and heat flux will be debatable or even undefinable for sufficiently small size. Therefore, besides the value of the heat flux varying along with the increasing temperature gradient, the applicable size of a heat conduction model should also be limited, which remains an open question.

In this work, it is found that this limitation of size can also be predicted by the nonlinear regime in the models with flux-limited behaviors, mainly including the phonon hydrodynamic [30,31] and Lagrange multiplier [32] models. For 1D steady-state heat conduction, where flux-limited behaviors are usually discussed, there will exist a critical size determined by the boundary temperatures, and the heat flux will exist only when the size is larger than the critical size. The critical sizes of these non-linear models can be regarded as the theoretical limits of their applicable ranges. The size and boundary effects for the existence of heat flux show different features from Fourier heat conduction, which can always guarantee the existence of heat flux for arbitrary boundary temperatures and size. It means that even in the limit of small heat flux (or small temperature gradient), these non-Fourier models with flux-limited behaviors will not reduce to Fourier's law and the nonlinear effects could not be negligible.

## 2. Critical size for heat flux in non-Fourier heat conduction

The flux-limited behaviors are mainly discussed for 1D steady-state boundary value problems in $[0,l]$ [25–29], where $T|_{x=0}=T_1$ and $T|_{x=l}=T_2$ (without loss of generality, $T_1<T_2$). In 1D steady-state problems, the heat flux the heat flux reduces to a constant scalar $q=-C$ and in consideration of $T_1<T_2$, only positive $C$ can satisfy the second law (the positive direction of the coordinate is from $x=0$ to $x=l$).

### 2.1. Phonon hydrodynamic model

We start from the phonon hydrodynamic model [30,31], which is derived from Callaway's relaxation approximation and maximum entropy principle

$$\boldsymbol{q}+\tau_R\frac{\partial\boldsymbol{q}}{\partial t}+\lambda\nabla T = -\tau_R\nabla\bullet\left(\frac{3v_g\langle\boldsymbol{qq}\rangle}{2v_gc_VT+\sqrt{4v_g^2c_V^2T^2-3\boldsymbol{q}^2}}\right), \tag{2}$$

where $\langle\boldsymbol{qq}\rangle$ is the deviatoric part of tensor $\boldsymbol{qq}$, $\tau_R$ is the relaxation time of phonon resistive scattering, $v_g$ is the average phonon group speed and $c_V$ is the heat capacity per unit volume. In 1D steady-state heat conduction, the governing equation of the phonon hydrodynamic model can be simplified to

$$C=\lambda\left[5-\frac{4}{\sqrt{1-\left(\frac{\sqrt{3}C}{2v_gc_VT}\right)^2}}\right]\frac{dT}{dx}. \tag{3}$$

In 1D steady-state problems, Eq. (3) is derived from Eq. (2), which can be found in Ref. [25] (see Eqs. (9)–(11) of Ref. [25]). From Eq. (3), it is obvious that the upper bound of the heat flux should be limited $|C|<\frac{2v_gc_VT}{\sqrt{3}}$ in mathematics. What's more, the second law of thermodynamics requires a non-negative effective thermal conductivity $\lambda[5-\frac{4}{\sqrt{1-(\frac{\sqrt{3}C}{2v_gc_VT})^2}}]\geq 0$, which will give a smaller upper bound $|C|\leq\frac{2\sqrt{3}}{5}v_gc_VT$. Similar upper bounds determined by $v_gc_VT$ can also be found in other models. The corresponding physical meaning is that the heat flux cannot be higher than the product of the energy density $c_VT$ and the maximum phonon speed $\sup(v_g)$. The relation between the boundary temperatures and heat flux can be given by the integration of Eq. (3)

$$5(T_2-T_1)-4\left[\sqrt{T_2^2-\left(\frac{\sqrt{3}C}{2v_gc_V}\right)^2}-\sqrt{T_1^2-\left(\frac{\sqrt{3}C}{2v_gc_V}\right)^2}\right] = \frac{Cl}{\lambda}. \tag{4}$$

In the cases of $C>0$, set $\frac{\sqrt{3}|C|}{2v_gc_V}=u_1$ $(0\leq u_1\leq T_1)$ and Eq. (4) is then rewritten as

$$5(T_2-T_1)-4\left(\sqrt{T_2^2-u_1^2}-\sqrt{T_1^2-u_1^2}\right)=\frac{2v_gc_Vl}{\sqrt{3}\lambda}u_1. \tag{5}$$

To determine the existence of $u_1$ in $[0,T_1]$, an auxiliary function is introduced as follows

$$f_1(u_1)=5(T_2-T_1)-4\left(\sqrt{T_2^2-u_1^2}-\sqrt{T_1^2-u_1^2}\right) -\frac{2v_gc_Vl}{\sqrt{3}\lambda}u_1, \tag{6}$$

whose first-order derivative is

$$\frac{df_1(u_1)}{du_1}=4u_1\left(\frac{1}{\sqrt{T_2^2-u_1^2}}-\frac{1}{\sqrt{T_1^2-u_1^2}}\right)-\frac{2v_gc_Vl}{\sqrt{3}\lambda}. \tag{7}$$

For $T_2>T_1$, we have $\frac{df_1(u_1)}{du_1}<0$ and hence, there is at most one solution. Due to $f_1(0)=(T_2-T_1)>0$, the existence of $u_1$ in $[0,T_1]$ needs

$$f_1(T_1)=5(T_2-T_1)-4\sqrt{T_2^2-T_1^2}-\frac{2v_gc_Vl}{\sqrt{3}\lambda}T_1\leq 0, \tag{8}$$

but inequality (8) is not necessarily satisfied, i.e., $\lim_{T_1\to 0}f_1(T_1)\to T_2>0$. In order to guarantee the existence of heat flux, the size should satisfy the following inequality

$$l\geq l_{c1}=\frac{\sqrt{3}\lambda}{2v_gc_V}\left[5\left(\frac{T_2}{T_1}-1\right)-4\sqrt{\frac{T_2^2}{T_1^2}-1}\right]. \tag{9}$$

When $1<\frac{T_2}{T_1}\leq\frac{41}{9}$, we find $l_{c1}\leq 0$ and therefore, inequality (9) always holds, which means that the heat flux of this case must exist. For $\frac{41}{9}<\frac{T_2}{T_1}$, $l_{c1}$ is positive and only when the size is larger than $l_{c1}$, the heat flux will exist. Accordingly, a size effect about the existence of heat flux is found for $\frac{41}{9}<\frac{T_2}{T_1}$. $l_{c1}$, which is determined by the ratio of boundary temperatures, can be regarded as a critical size for the existence of heat flux. This critical size gives a theoretical maximum applicable range for the phonon hydrodynamic model, which also shows the boundary effects on the existence of heat flux. If the ratio of the boundary temperatures is sufficiently small, the phonon hydrodynamic model could be applied to small (even arbitrarily small for $\frac{T_2}{T_1} \le \frac{41}{9}$) scale heat conduction problems in theory. As the ratio increases, the critical size will increase and the theoretically applicable range of the phonon hydrodynamic model also becomes narrower. When the ratio tends to infinity, the critical size will also tend to infinity, which seems that this model is inappropriate to the case of $\frac{T_2}{T_1} \gg 1$. On the other hand, for 1D heat conduction with a fixed size, the ratio of the boundary temperatures will be limited by the size and in small scale problems, the ratio cannot be very large. It should be pointed that the above conclusions are based on constant physical properties. It requires that the physical properties display vanishingly small variations with the temperature. However, this requirement can only be satisfied in narrow temperature ranges in many cases. Therefore, our conclusions might not be applicable for large $\frac{T_2}{T_1}$ in these cases. Non-positive length for $1 < \frac{T_2}{T_1} \le \frac{41}{9}$ is not physically meaningful, but the negative values of $l_{c1}$ can show a mathematical possibility for $C < 0$. As we have mentioned, the positive direction of the coordinate is from $x = 0$ to $x = l$ and then, the temperature difference is in the positive direction of the coordinate. According to the second law, the heat flux must be in the negative direction of the coordinate $q = -C < 0$ $(C > 0)$. Therefore, $C < 0$ means that certain possible violation of the second law might be predicted. In this case, $u_1$ satisfies the following equation

$$5(T_2 - T_1) - 4\left(\sqrt{T_2^2 - u_1^2} - \sqrt{T_1^2 - u_1^2}\right) + \frac{2 v_g c_V l}{\sqrt{3}\lambda} u_1 = 0, \quad (10)$$

and its corresponding auxiliary function is rewritten as

$$f_2(u_1) = 5(T_2 - T_1) - 4\left(\sqrt{T_2^2 - u_1^2} - \sqrt{T_1^2 - u_1^2}\right) + \frac{2 v_g c_V l}{\sqrt{3}\lambda} u_1. \quad (11)$$

It has been mentioned that $l_{c1}$ is negative for $1 < \frac{T_2}{T_1} < \frac{41}{9}$, since it is possible that $f_2(T_1) = \frac{2 v_g c_V T_1}{\sqrt{3}\lambda}(l + l_{c1}) < 0$ as long as $l$ is sufficiently small $l < |l_{c1}|$. Due to $f_2(0) = (T_2 - T_1) > 0$, there must exist at least one $u_1$ satisfying Eq. (10). This means that neither existence nor uniqueness are guaranteed by the phonon hydrodynamic model for the problem with sufficiently small scale, which could also cause the mathematical possibility of violating the second law. If this mathematical possibility must be eliminated, we can let $l > -\min(l_{c1})$, which will give another critical size

$$l > l_{c1}^* = -\min(l_{c1}) = \frac{\sqrt{3}\lambda}{6 v_g c_V}. \quad (12)$$

The mean free path of heat carriers could usually be estimated as $L \cong \frac{\lambda}{3 v_g c_V}$, and we have $l_{c1}^* = 0.866L$. It seems that the phonon hydrodynamic model could be applicable to heat conduction processes whose sizes are comparable to (or even slightly smaller than) the mean free path of heat carriers. If $l \ll L$, this model will predict several theoretical problems including multiplicity and violation of the second law. To sum up, the finally critical size of the phonon hydrodynamic model can be written as

$$l_{PH} = \max\left(l_{c1}^*, l_{c1}\right) = \max\left(|l_{c1}|\right). \quad (13)$$

As comparison, non-linear Fourier heat conduction with the thermal conductivity expressed as $\lambda = \lambda(T)$ can always guarantee the existence, uniqueness and the second law for 1D steady-state problems with arbitrary boundary temperatures and sizes. These differences mean that the phonon hydrodynamic model cannot reduce to Fourier's law even in the limit of small heat flux, which shows a different conclusion from Ref. [25].

### 2.2. Nonequilibrium thermodynamics models

Next, we focus on three nonequilibrium thermodynamic models which are obtained in the spirit of extended irreversible thermodynamics (EIT) [6]. The first is the Lagrange multiplier model [32], which is on the basis of the Gibbs relation and information theory

$$\boldsymbol{q} = -\frac{\lambda}{2}\left[1 - \frac{3}{2}\left(\frac{\boldsymbol{q}}{v_g c_V T}\right)^2 + \sqrt{1 - \frac{3}{4}\left(\frac{\boldsymbol{q}}{v_g c_V T}\right)^2}\right]\nabla T. \quad (14)$$

Similar to the above case, the size also needs to satisfy an inequality for the existence of heat flux

$$l \ge l_{c2} = \frac{\sqrt{3}\lambda}{4 v_g c_V}\left[\frac{T_2}{T_1} + \frac{2T_1}{T_2} - 3 + \sqrt{\frac{T_2^2}{T_1^2} - 1} - \arccos\left(\frac{T_1}{T_2}\right)\right]. \quad (15)$$

Let $\arccos\left(\frac{T_1}{T_2}\right) = \theta$ $(0 < \theta < \frac{\pi}{2})$ and we have

$$l_{c2}(\theta) = \frac{\sqrt{3}\lambda}{4 v_g c_V}\left(2\cos\theta + \frac{1}{\cos\theta} - 3 + \tan\theta - \theta\right), \quad (16)$$

which is not necessarily positive. There exists a unique $\theta_c \in (0, \frac{\pi}{2})$ satisfying $l_{c2}(\theta_c) = 0$. For $\cos\theta_c \le \frac{T_1}{T_2} < 1$, its corresponding heat flux will always exist, while for $0 < \frac{T_1}{T_2} \le \cos\theta_c$, there is also a critical size $l_{c2}(\theta)$ determined by the ratio of boundary temperatures. Then, we can conclude similar size and boundary effects on the existence of heat flux to the phonon hydrodynamic model. What's more, the Lagrange multiplier model can predict the mathematical possibility for $C < 0$ either, and the critical size for avoiding this possibility can be given through the same method

$$l > l_{c2}^* = -\min(l_{c2}) = \frac{\sqrt{3}\lambda}{4 v_g c_V}\left(3 + \frac{\pi}{6} - 2\sqrt{3}\right) = 0.026L. \quad (17)$$

Compared with the phonon hydrodynamic model, the Lagrange multiplier model predicts a much smaller critical size, which seems that this model has larger theoretically applicable range than the phonon hydrodynamic model. However, the phonon hydrodynamic model is selected as a credible physical standard in Ref. [18] for its more rigorous physical foundation and from this perspective, it should be more universal than the Lagrange multiplier model. Thus, the actual applicable range of the Lagrange multiplier model might be much narrower than its theoretically applicable range.

The second is the hierarchy moment model [33] which incorporates an infinite hierarchy of moments in the framework of EIT

$$\boldsymbol{q} = -\frac{\lambda \nabla T}{\frac{1}{2} + \sqrt{\frac{1}{4} + L^2\left(\frac{\nabla T}{T}\right)^2}}. \quad (18)$$

In 1D steady-state problems, we can obtain

$$\frac{Cl}{\lambda(T_2 - T_1)} + \frac{C^2 L^2}{\lambda^2 T_2 T_1} = 1, \quad (19)$$

which is a monadic quadratic equation with a positive solution and a negative solution. In the cases of $C < 0$, we have $\frac{dT}{dx} < 0$ and $T|_{x=0} < T|_{x=l}$, which is in contradiction with $T_1 < T_2$. Thus, $C$

must be positive and hence the second law will always be satisfied. In summary, besides obeying the second law in physics, the hierarchy moment model also guarantees the uniqueness and existence independent of size in mathematics. The third model is the nonlinear phonon hydrodynamic model [26,27] which is obtained by introducing dynamical nonequilibrium temperature

$$\boldsymbol{q}+\tau_R\frac{\partial \boldsymbol{q}}{\partial t}+\lambda\nabla T=\frac{2\tau_R}{Tc_V}\boldsymbol{q}\bullet\nabla\boldsymbol{q}+l^2\left[\nabla^2\boldsymbol{q}+2\nabla(\nabla\bullet\boldsymbol{q})\right]. \tag{20}$$

The nonlinear phonon hydrodynamic model will reduce to Fourier heat conduction in 1D steady-state problems [25], $\lambda\frac{dT}{dx}=C$, whose uniqueness and existence are independent of the size. Different from the phonon hydrodynamic and Lagrange multiplier models, the size and boundary effects on the existence of heat flux are not found in these two models.

### 2.3. Phenomenological models

Phenomenological method is another important type of nonlinear heat conduction with flux-limited behaviors. The tempered diffusion model [34] is a typical one

$$\boldsymbol{q}=-\lambda\sqrt{1-\left(\frac{\boldsymbol{q}}{v_g c_V T}\right)^2}\frac{dT}{dx}, \tag{21}$$

and the existence of heat flux in 1D steady-state problems needs

$$l\geq l_{c3}=\frac{\lambda}{v_g c_V}\left[\sqrt{\frac{T_2^2}{T_1^2}-1}-\arccos\left(\frac{T_1}{T_2}\right)\right]. \tag{22}$$

Unlike the cases of the phonon hydrodynamic and Lagrange multiplier models, $l_{c3}(\theta)=\frac{\lambda}{v_g c_V}(\tan\theta-\theta)$ is always positive. Accordingly, for any problem with fixed boundary temperatures, there always exists a critical size $l_{c3}$ for the existence of heat flux. On the other hand, the positivity of $l_{c3}$ guarantees the uniqueness and second law.

The second phenomenological model is the generalized nonlinear model [29] which is obtained by introducing dynamical nonequilibrium temperature

$$\begin{aligned}&\boldsymbol{q}+\tau_R\frac{\partial \boldsymbol{q}}{\partial t}+\lambda\left(1+\beta\boldsymbol{q}^2\right)\nabla T\\&\quad=\mu\boldsymbol{q}\bullet\nabla\boldsymbol{q}+\mu'\nabla\boldsymbol{q}\bullet\boldsymbol{q}+L^2\left[\nabla^2\boldsymbol{q}+2\nabla(\nabla\bullet\boldsymbol{q})\right],\end{aligned} \tag{23}$$

where $\beta$, $\mu$ and $\mu'$ are the phenomenological coefficients. In 1D steady-state problems, this model also gives a monadic quadratic equation about $C$ as follows [25]

$$1+\beta C^2=\frac{Cl}{\lambda(T_2-T_1)}. \tag{24}$$

When $\beta<0$, Eq. (23) will always have a positive solution and a negative solution. It has been pointed that the case of $\beta>0$ will lead to unphysical behaviors of heat flux [29]. From the viewpoint of existence, positive $\beta$ makes $\frac{l^2}{\lambda^2(T_2-T_1)^2}<4\beta$ possible and there will be another critical size for existence

$$l\geq l_{c4}=2\lambda(T_2-T_1)\sqrt{\beta}. \tag{25}$$

For $l>l_{c4}$, there are two possible solutions

$$C_\pm=\frac{l}{2\lambda(T_2-T_1)\beta}\pm\sqrt{\frac{l^2}{4\lambda^2(T_2-T_1)^2\beta^2}-\frac{1}{\beta}}, \tag{26}$$

which are both positive and then, the second law can be satisfied for each case. As the boundary temperature difference $T_2-T_1$ increases, $C_+$ decreases while $C_-$ increases. When $(T_2-T_1)\to 0^+$, $C_+$ even tends to infinity, which seems unphysical and therefore, it is better to select $C_-$ as the physically meaningful solution. The thermon gas model [21–24] is another widely discussed phenomenological model derived from Einstein's mass–energy relation and phonon hydrodynamics

$$\boldsymbol{q}+\tau_T\frac{\partial \boldsymbol{q}}{\partial t}+\lambda\nabla T=-\tau_T\nabla\bullet\left(\frac{\boldsymbol{qq}}{c_V T}\right), \tag{27}$$

where $\tau_T=\frac{\rho\tau_R v_g^2}{6\gamma c_V T}$ is the relaxation time of thermon gas, $\rho$ is the mass density, and $\gamma$ is the Grüneisen constant. It also predicts a monadic quadratic equation about $C$

$$C^2\left(\frac{\rho}{4\gamma c_V^3 T_1^2}-\frac{\rho}{4\gamma c_V^3 T_2^2}\right)+(T_1-T_2)=\frac{Cl}{\lambda}, \tag{28}$$

which will also have a positive solution and a negative solution. Eq. (28) is derived from the integral of the 1D constitutive relation, which can be found in Ref. [25] (see Eq. (16) of Ref. [25]). Compared with the tempered diffusion model, the last two phenomenological models can also guarantee the existence independent of size but they could predict the multiplicity and mathematical possibility of violating the second law.

## 3. Non-linear Fourier heat conduction with flux-limited behaviors

It should be noted that Fourier heat conduction with the thermal conductivity $\lambda=\lambda(T)$ can also predict flux-limited behaviors. The fast diffusion [14], which is usually applied to complex physical systems, is a common type, where $\lambda=\frac{\lambda_0}{T^\alpha}$ ($\lambda_0$ is a positive constant). In 1D steady-state problems, heat flux always exists for arbitrary boundary temperatures and size

$$C=\begin{cases}\frac{\lambda_0}{(1-\alpha)l}\left(T_2^{1-\alpha}-T_1^{1-\alpha}\right), & \alpha\neq 1\\ \frac{\lambda_0}{l}\ln\frac{T_2}{T_1}, & \alpha=1.\end{cases} \tag{29}$$

Eq. (29) shows that predicting a limited flux as the temperature difference tends to infinity needs $\alpha>1$. The saturation heat flux of fast diffusion is $\frac{\lambda_0 T_1^{1-\alpha}}{(\alpha-1)l}$, which will decrease with $T_1$ increasing. In contrast, the saturation heat fluxes of the above non-Fourier models are increasing with $T_1$ increasing (usually proportional to $T$). In unsteady-state problems, this singular type of thermal conductivity $\lambda=\frac{\lambda_0}{T^\alpha}$ will reveal abundant original physicomathematical phenomenon, which is quite different from the cases of constant thermal conductivity. One remarkable behavior is superfast diffusion, a special subclass paired with diffusion choking a finite time where a heat conduction process could terminate within a finite time. What's more, the type of fast diffusion could predict travelling wave solutions in the forms of $T_1(x-Ut)$ and $T_2(x+Ut)$. The superposition of these travelling wave solutions $T(x,t)=T_1(x-Ut)+T_2(x+Ut)$ will also satisfy the governing equation, which is exactly a well-known wave-like behavior predicted by the linear hyperbolic equation $\frac{\partial^2 T}{\partial t^2}=U^2\nabla^2 T$.

It seems that Fourier's law is more robust than other models, and we will discuss why and how to achieve similar robustness. The effective thermal conductivities in this work can be summarized as $\lambda_{eff}=\lambda_{eff}(T,\boldsymbol{q})$. In 1D steady-state problems, we have $C=\lambda_{eff}(T,-C)\frac{dT}{dx}$, whose integral gives

$$\begin{aligned}&Cl=\int_{T_1}^{T_2}\lambda_{eff}(T,-C)dT \quad\Leftrightarrow\\&l=\frac{1}{C}\int_{T_1}^{T_2}\lambda_{eff}(T,-C)dT=\Psi(T_1,T_2,C).\end{aligned} \tag{30}$$

The well-posedness for the heat flux needs a positive and unique $C$ satisfying the above equation. For the uniqueness, $\Psi(T_1, T_2, C)$ must be monotonous for arbitrary boundary temperatures ($T_2 > T_1 > 0$) and $C > 0$, $\frac{\partial \Psi(T_1,T_2,C)}{\partial C} \neq 0$. To guarantee the existence for arbitrary $l \in (0, +\infty)$, we need $\sup[\Psi(T_1, T_2, C)] = +\infty$ and $\inf[\Psi(T_1, T_2, C)] = 0$. Generally speaking, $\lambda_{eff}$ should be positive when $C \to 0$ and hence we have $\lim_{C\to 0} \Psi(T_1, T_2, C) = \lim_{C\to 0} \frac{1}{C}\int_{T_1}^{T_2} \lambda_{eff}(T, 0)dT = +\infty$. Then, $\sup[\Psi(T_1, T_2, C)] = +\infty$ can be satisfied and because of the monotonicity, $\frac{\partial \Psi(T_1,T_2,C)}{\partial C} \leq 0$. According to $\frac{\partial \Psi(T_1,T_2,C)}{\partial C} \leq 0$ and $\inf[\Psi(T_1, T_2, C)] = 0$, we can give $\lim_{C\to+\infty} \Psi(T_1, T_2, C) = 0$. In summary, a well-posed model need to satisfy $\frac{\partial \Psi(T_1,T_2,C)}{\partial C} \leq 0$ and $\lim_{C\to+\infty} \Psi(T_1, T_2, C) = 0$ for arbitrary $T_2 > T_1 > 0$. For Fourier's law, the effective thermal conductivity is independent of the heat flux $\lambda_{eff} = \lambda_{eff}(T)$ and $\Psi(T_1, T_2, C) = \frac{1}{C}\int_{T_1}^{T_2} \lambda_{eff}(T)dT$. It is not difficult to find that the above two conditions $\frac{\partial \Psi(T_1,T_2,C)}{\partial C} \leq 0$ and $\lim_{C\to+\infty} \Psi(T_1, T_2, C) = 0$ will be satisfied for arbitrary $T_2 > T_1 > 0$. That is why Fourier's law is so robust in 1D steady-state problems. However, in multiple-dimensional problems, Fourier's law would not be well-posed either. As we have mentioned, the type of fast diffusion $\lambda_{eff} \propto \frac{1}{T^\alpha}$ will results in no physically meaningful solutions for $\alpha > 1$ [14], which is caused by the singularity $\lim_{T\to 0^+} \lambda_{eff} = +\infty$. Thus, the robustness requires at least three conditions ($T_2 > T_1 > 0$), $\lim_{T\to 0^+} \lambda_{eff} = +\infty$, $\frac{\partial \Psi(T_1,T_2,C)}{\partial C} \leq 0$ and $\lim_{C\to+\infty} \Psi(T_1, T_2, C) = 0$. Now we consider the strong nonlinear case of the generalized nonlinear model Eq. (23), where nonlinear term $\lambda\beta \boldsymbol{q}^2 \nabla T$ plays a leading role. In this the strong nonlinear case, the linear term ($\lambda \nabla T$), the nonlocal terms ($\mu \boldsymbol{q} \bullet \nabla \boldsymbol{q}$, $\mu' \nabla \boldsymbol{q} \bullet \boldsymbol{q}$, $L^2 \nabla^2 \boldsymbol{q}$, $2L^2 \nabla(\nabla \bullet \boldsymbol{q})$) and the relaxation term $\tau_R \frac{\partial \boldsymbol{q}}{\partial t}$ are neglected, and then Eq. (23) will reduce to

$$\boldsymbol{q} \cong \lambda\beta \boldsymbol{q}^2 \nabla T. \tag{31}$$

The strong nonlinearity will emerge when the physical properties satisfy $\beta \boldsymbol{q}^2 \gg 1$ and $\tau_R, \mu', \mu, L \to 0$. In 1D problems, we can let $u = \boldsymbol{q}^{-1}$ and the governing equation is subsequently rewritten as

$$\frac{\partial u}{\partial t} = \frac{1}{c_V} \nabla \left( -\frac{\lambda\beta}{u^2} \nabla u \right), \tag{32}$$

which is also a fast diffusion type ($\beta < 0$). With other non-Fourier effects including nonlocality and relaxation neglected, non-Fourier characteristics caused by nonlinearity can be reflected. Eq. (32) shows that the nonlinear non-Fourier effects would predict similar behaviors to fast diffusion.

## 4. Conclusions

Size and boundary effects on the existence of heat flux are found for several non-linear models with flux-limited behaviors including the phonon hydrodynamic, Lagrange multiplier and tempered diffusion models. In 1D steady-state problems, the existence of heat flux needs the size larger than certain critical size, which is determined by the ratio of boundary temperatures. For the phonon hydrodynamic and Lagrange multiplier models, the critical sizes only exist for certain boundary temperatures while for the tempered diffusion model, there always exists a critical size. These size and boundary effects provide theoretical rough estimations for the applicable ranges of the models. The second law will always be satisfied by the tempered diffusion model for its non-negative form of the effective thermal conductivity. In contrast, for the problems with sufficiently small size, the phonon hydrodynamic and Lagrange multiplier models could predict the mathematical possibility of violating the second law, which could be avoided as long as the size is larger than the critical sizes.

From the physical point of view, for any non-zero temperature difference, there must exist a real value of the heat flux. However, it is found that several models could not predict real heat flux in some problems. To satisfy the constitutive relations of these models, the heat flux might need imaginary part in some cases, which seems unphysical. Therefore, the constitutive relations are not physically applicable in these cases, which are related to the scales. For given boundary conditions and physical properties, there will exist a critical length scale. As long as the scale of a heat conduction problem is smaller than this critical length scale, the heat flux predicted by the constitutive models will contain imaginary part. It means that the heat conduction models are not physically applicable for the problem with "sufficiently small scale". "Sufficiently small" means that the scale of this problem needs to be smaller than the critical length scale. The constitutive relations of these heat conduction models are expressed by the macroscopic temperature, heat flux and their derivatives. Strictly speaking, the definitions of these macroscopic quantities need local equilibrium, which would be debatable with the scale tending to zero. For instance, when the scale is comparable to the molecular or atomic diameter, it is not appropriate to define a continuous temperature distribution or spatial differentials. As the definitions of the macroscopic quantities are not well-defined (or even undefinable), these heat conduction models will not be physically applicable either. Thus, the physically applicable ranges of the models must be limited by the scale, which need sufficiently large scales to define the macroscopic quantities. On the other hand, the non-existence of the real heat flux also gives mathematically applicable ranges, which also need scales larger than the critical sizes. It is found that similar requirements for applicable ranges of the heat conduction models are established both on physics and mathematics. The critical size of a model is its mathematical maximum limit, and the corresponding physically applicable range should have stronger limit. The critical size could also help understand flux-limited behaviors when the mean thermal conductivity $\bar{\lambda}$ satisfies $\lim_{l\to 0} \bar{\lambda} \neq 0$. In 1D heat conduction, we have $|\boldsymbol{q}| = \bar{\lambda}\frac{(T_2-T_1)}{l}$ and for a given temperature difference $(T_2 - T_1)$, $|\boldsymbol{q}|$ will increases with decreasing $l$. If $\lim_{l\to 0} \bar{\lambda} \neq 0$, the heat flux $|\boldsymbol{q}| = \bar{\lambda}\frac{(T_2-T_1)}{l}$ will tend to infinity as $l \to 0$. Therefore, if the heat flux has an upper bound $\sup(|\boldsymbol{q}|) < +\infty$, $l$ cannot be arbitrarily small $l \geq \bar{\lambda}\frac{(T_2-T_1)}{\sup(|\boldsymbol{q}|)} > 0$. Then, the positive lower bound $\bar{\lambda}\frac{(T_2-T_1)}{\sup(|\boldsymbol{q}|)}$ could be understood as the critical size, which also shows why the critical size is effected by the boundary temperatures. It is found that the phonon hydrodynamic and Lagrange multiplier models could predict multiple solutions in a given 1D steady-state problem. Some of the solutions correspond to the cases of $C < 0$. As the boundary conditions are taken as $T_1 = T|_{x=0} < T|_{x=l} = T_2$, the temperature difference is in the positive direction of the coordinate. According to the second law, the heat flux must be in the negative direction of the coordinate $q = -C < 0 \Leftrightarrow C > 0$. Therefore, $C < 0$ means that heat could spontaneously flow from the cold boundary to the hot boundary, which seems to violate the second law. Note that the unphysical direction of heat transfer is found in steady-state problems. This violation of Clausius statement is not only in a local region or transient moment but also in global heat transfer. In summary, the phonon hydrodynamic and Lagrange multiplier models will predict multiple solutions, and some of the solutions have unphysical direction of heat transfer, which violates the second law. One interesting conclusion is that similar to the non-existence of the heat flux, the multiplicity paired with the possible violation of the second law is also related to the scale. As shown in inequalities (12) and (17), the multiplicity paired with the possible violation of the second law could be avoided when the two inequalities are satisfied. The supplementary critical sizes ($l_{c1}^*$ and $l_{c2}^*$) might give other applicable ranges for the heat conduction models.

These size and boundary effects are not found in the nonlinear phonon hydrodynamic, thermon gas, hierarchy moment and generalized nonlinear ($\beta < 0$) models. The generalized nonlinear and thermon gas models will predict the possibility of multiplicity for any size while the hierarchy moment and nonlinear phonon hydrodynamic models are always well-posed for 1D steady-state heat conduction. However, the phonon hydrodynamic model, which is considered as a credible physical standard in Ref. [25] for its rigorous physical foundation, should be more universal than other models. It seems that size and boundary effects predicted by the phonon hydrodynamic model also exist for other models, and the applicable ranges must be limited for all the models. Besides these non-Fourier models, Fourier heat conduction in the type of fast diffusion could also predict flux-limited behaviors. Compared with the non-Fourier models, the type of fast diffusion has at least three different features: existence independent of size, decreasing saturation heat flux with increasing temperature and dimensional dependency. The flux-limited behaviors in heat conduction is usually discussed in 1D simplified problems, and $l$ is the size in the heat transfer direction. Strictly speaking, it requires that there should exist a main heat transfer direction, and heat transfer in other directions could be neglected. Therefore, $l$ need to be selected on the basis of specific heat conduction problems. In nanoscale heat transport, where the models are used to describe the non-Fourier effects, $l$ is usually the thickness of a nanoscale thin film or the length of a nanowire. For more universal cases, i.e., variable cross-section, the generalization or selection of $l$ needs further discussion. If the heat conduction models predict obvious non-Fourier effects only at small scale, $l$ can be selected as the minimum size of the region. However, it has been shown that the models can predict obvious non-Fourier effects at large scale. For instance, the non-Fourier effects of the phonon hydrodynamic model are also influenced by boundary temperatures, which will be obvious as long as $T_2 \gg T_1$. Accordingly, for these cases, the boundary conditions should also be considered in the selection of $l$, which is different from the selection of the "characteristic length scale" in Fourier heat conduction. Generally speaking, when the models reduce to Fourier's law, $l$ should also reduce to the "characteristic length scale" used in Fourier heat conduction. Our study on flux-limited behaviors is only for the 1D cases, while flux-limited behaviors also exist in multi-dimensional problems [35], which need further discussion.

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